Problem Sheet 1

Problem 1

Fix $n \geq 1$ and $\zeta \in \mathbb{C}$, a primitive n-th root of unity. Let $\mathbb{Q}(\zeta) \subseteq \mathbb{C}$ be the subfield generated by ζ. It is a splitting field of $T^n - 1$, hence Galois; denote by $G := \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ its Galois group.

(a) Show that for every $\sigma \in G$, there is a unique $e(\sigma) \in (\mathbb{Z}/n\mathbb{Z})^\times$ such that $\sigma(\zeta) = \zeta^{e(\sigma)}$ and that the assignment $\sigma \mapsto e(\sigma)$ defines a group isomorphism

$$G \xrightarrow{\cong} (\mathbb{Z}/n\mathbb{Z})^\times.$$

(b) Determine the Galois group and all intermediate fields of $\mathbb{Q}(\zeta_8)/\mathbb{Q}$, where ζ_8 is a primitive 8-th root of unity.

Problem 2

Fix $n \geq 1$ and let K be a field that contains a primitive n-th root of unity ζ. Let L/K be a Galois extension with Galois group G cyclic of order n. The following will show that L has the simple form $K(\sqrt[n]{a})$ for some $a \in K^\times$.

(a) Fix a generator $\sigma \in G$ and view it as an endomorphism of the K-vector space L. Show that its only possible eigenvalues are the n-th roots of unity ζ^i, $i \in \mathbb{Z}/n\mathbb{Z}$, and that L fully decomposes into eigenspaces,

$$L = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} L_i,$$

where $L_i = \{\ell \mid \sigma(\ell) = \zeta^i \ell\}$. Prove that $L_i L_j \subseteq L_{i+j}$ holds.

(b) Show that $\dim L_i = 1$ for all i. Conclude that there is an element $0 \neq \alpha \in L$ such that $\sigma(\alpha) = \zeta\alpha$, that such an element generates L and that it satisfies $\alpha^n \in K$.

Problem 3

An integral domain A is *euclidean* if there is a degree map $\deg\colon A \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0}$ such that for all $x, y \in A, y \neq 0$ there are $q, r \in A$ such that

$$x = qy + r \quad \text{with } r = 0 \text{ or } \deg(r) < \deg(y).$$

(a) Show that a euclidean ring is a PID (principal ideal domain).

(b) Show that $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$ are euclidean.